JOURNAL OF FUNCTIONAL ANALYSIS 6, 318–346 (1970)

On the Repeated Measurement of Continuous Observables in Quantum Mechanics

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Communicated by Irving Segal

Received September 12, 1969

We generalize Umegaki's definition of an expectation on the algebra of all bounded operators on a Hilbert space, and classify certain classes of expectations subject to a covariance condition with respect to a unitary representation of a given locally compact group. While Umegaki's expectations only exist for discrete observables, we show that interesting classes of our expectations exist in the general case of continuous observables, and discuss the implications of our work in the theory of quantum measurements.

1. INTRODUCTION

In formulating a mathematical theory to describe the process of making repeated observations on each sample of a statistical ensemble in quantum mechanics one of the central notions is that of an expectation, which we now define. If X is a Borel space, $\mathscr{H}$ is a Hilbert space and $\mathscr{L}(\mathscr{H})$ is the von Neumann algebra of all bounded operators on $\mathscr{H}$ an *expectation* is a function $\mathscr{E}^*(E, A)$ defined for all Borel sets $E \subseteq X$ and all $A \in \mathscr{L}(\mathscr{H})$ with values in $\mathscr{L}(\mathscr{H})$. $\mathscr{E}^*$ is supposed to have the following properties.

(i) $A \geqslant 0$ implies $\mathscr{E}^*(E, A) \geqslant 0$ for all $E \subseteq X$.

(ii) $A \to \mathscr{E}^*(E, A)$ is linear for all $E \subseteq X$.

(iii) $E \to \mathscr{E}^*(E, A)$ is σ-additive for all $A \in \mathscr{L}(\mathscr{H})$ the sum converging in the weak operator topology.

* At present on leave from the University of Oxford. Work supported by N.S.F. Grant GP-7952X and A.F.O.S.R. Contract No. F44620-67-C-0029.

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(iv) $\mathscr{E}^*$ is normal. That is, if A_n is a monotone net in $\mathscr{L}(\mathscr{H})$ converging weakly to A then $\mathscr{E}^*(E, A_n)$ converges monotonely and weakly to $\mathscr{E}^*(E, A)$ for all $E \subseteq X$.
We say that $\mathscr{E}^*$ is proper if also

(v) $\mathscr{E}^*(X, I) = I$.

The only class of expectations studied at all seriously so far are constructed from the conditional expectations in the sense of Umegaki [1, 2, 3]. If $P(\cdot)$ is a projection-valued measure on X and T is a normal conditional expectation in the sense of Umegaki with domain $\mathscr{L}(\mathscr{H})$ and range the commutant of $\{P(E) : E \subseteq X\}$ then the formula

$$\mathscr{E}^*(E, A) = P(E)\, T(A)$$

defines a proper expectation in our sense. Unfortunately if the projection-valued measure is totally non-atomic it is known that such a conditional expectation T does not exist, so this approach is unsuccessful. However the problem is an important one and we provide an alternative approach applicable to the case where X is not a discrete Borel space below. In particular we construct and classify certain families of expectations subject to the following restraint. We suppose X is a transitive G-space where G is a separable locally compact topological group, that G has a strongly continuous unitary representation U on $\mathscr{H}$ and that $\mathscr{E}^*$ is *covariant* in the sense that

$$\mathscr{E}^*(E_{g^{-1}}, A) = U_g \mathscr{E}^*(E, U_g{}^* A U_g)\, U_g{}^*$$

for all $g \in G$, $E \subseteq X$ and $A \in \mathscr{L}(\mathscr{H})$.

In Section 3 we classify all covariant expectations subject to the restriction that $\mathscr{H}$ is finite-dimensional.

In Section 4 we classify all covariant expectations on a separable Hilbert space subject to the restriction that $\mathscr{E}^*(E, I)$ is a projection for all $E \subseteq X$. This very strong condition is important since expectations constructed from a conditional expectation in the sense of Umegaki would have this property. However there always do exist expectations with this property.

In Section 5 we construct a class of covariant expectations such that the operators $\mathscr{E}^*(E, I)$, as E runs over all Borel sets in X, lie in an abelian von Neumann algebra. The expectations in Section 4 are quite different from these and are not limiting cases of these expectations in any sense.

In Section 6 we study and compare the properties of the expectations obtained. In particular we show how the failure of the Umegaki conditional expectation approach is linked to the Heisenberg uncertainty principle.

2. Preliminaries

We shall continue with the notation and conventions of [4] and refer the reader there for the physical background to the definition of an instrument. (V, τ) will denote a state space and after this section we shall consider only the main case of interest to quantum mechanics, where V is the ordered Banach space $\mathscr{T}_s(\mathscr{H})$ of all self-adjoint trace class operators on a separable Hilbert space $\mathscr{H}$, and where τ is the usual trace on $\mathscr{T}_s(\mathscr{H})$. If X is a separable locally compact Hausdorff space and (V, τ) a state space, an instrument $\mathscr{E}$ on X can be defined in three possible ways:

(D1) An instrument is a positive σ-additive measure $\mathscr{E}$ on X with values in $\mathscr{L}(V, V)$ such that for all A in V we have $\tau(\mathscr{E}_X A) = \tau(A)$.

(D2) An instrument is a bilinear map

$$\mathscr{E}' : B(X) \times V \to V$$

where $B(X)$ is the space of real bounded Borel functions on X, such that

(i) if $f \in B(X)^+$ and $A \in V^+$ then $\mathscr{E}'(f, A) \in V^+$;

(ii) if $f_n \nearrow f$ pointwise in $B(X)$ and $A \in V^+$ then $\mathscr{E}'(f_n, A) \to \mathscr{E}'(f, A)$ in norm;

(iii) $\tau[\mathscr{E}'(1, A)] = \tau(A)$ for all A in V.

(D3) An instrument is a bilinear map

$$\mathscr{E}'' : K(X) \times V \to V$$

where $K(X)$ is the space of real continuous functions of compact support of X, such that

(i) if $f \in K(X)^+$ and $A \in V^+$ then $\mathscr{E}''(f, A) \in V^+$;

(ii) if $f_n \nearrow 1$ pointwise in $K(X)$ and $A \in V^+$ then $\tau[\mathscr{E}''(f_n, A)] \nearrow \tau(A)$.

We leave the reader to verify, using the methods of [4, Theorem 1] that these definitions are equivalent in the sense that any instrument of the type $\mathscr{E}$ or $\mathscr{E}''$ has a unique extension to an instrument of the type $\mathscr{E}'$; we will not distinguish between these different definitions from now on. The notion of an instrument is dual to the notion of an expectation in the sense that if $\mathscr{E}$ is an instrument on X there is a unique expectation $\mathscr{E}^*$ on X such that

$$\operatorname{tr}[\mathscr{E}(E, A)\, B] = \operatorname{tr}[A\mathscr{E}^*(E, B)]$$

for all $E \subseteq X$, $A \in V$ and $B \in \mathscr{L}(\mathscr{H})$. Conversely every expectation is the dual of a unique instrument. Every instrument $\mathscr{E}$ determines a unique observable $A(\cdot)$ in the sense of [4] and $A(\cdot)$ is given in terms of $\mathscr{E}^*$ by $A(E) = \mathscr{E}^*(E, I)$ for all Borel sets $E \subseteq X$.

Now let us suppose that G is a separable locally compact group and that X is a transitive G-space with G acting on the right as in [5]. Let us also suppose that G acts as a strongly continuous group of positive τ-automorphisms of V; in other words G is a strongly continuous group of automorphisms of V such that if $s \in G$ and $A \in V^+$ then $sA \in V^+$ and $\tau(sA) = \tau(A)$. Then the main problem of this paper is that of classifying some of the instruments on X to $\mathscr{L}(V, V)$ satisfying

$$\mathscr{E}(sf, A) = s[\mathscr{E}(f, s^{-1}A)]$$

for all s in G, f in $K(X)$ and A in V. This is merely a covariance condition demanding that the instrument $\mathscr{E}$ respect the group G of symmetries of the physical system.

It was shown in [4] that corresponding to every instrument $\mathscr{E}$ there is a natural observable $\mathscr{A}_{\mathscr{E}}$ which according to [4] is a positive linear map $\mathscr{A}_{\mathscr{E}} : K(X) \to V^*$ defined by

$$\langle \mathscr{A}_{\mathscr{E}}(f), A\rangle = \tau[\mathscr{E}(f, A)]$$

for all f in $K(X)$ and all A in V. There is a dual representation of G as a weakly continuous group of positive automorphisms of V^* mapping the identity to the identity and given by

$$\langle sA, B\rangle = \langle A, s^{-1}B\rangle$$

for all s in G, A in V and B in V^*. If $\mathscr{E}$ satisfies the above covariance condition then for all s in G, f in $K(X)$ and A in V

$$\begin{aligned}\langle \mathscr{A}_{\mathscr{E}}(sf, A\rangle &= \tau[\mathscr{E}(sf, A)] = \tau[s\mathscr{E}(f, s^{-1}A)]\\ &= \tau[\mathscr{E}(f, s^{-1}A)] = \langle \mathscr{A}_{\mathscr{E}}(f), s^{-1}A\rangle = \langle s\mathscr{A}_{\mathscr{E}}(f), A\rangle\end{aligned}$$

so that $\mathscr{A}_{\mathscr{E}}(sf) = s\mathscr{A}_{\mathscr{E}}(f)$.

If $\mathscr{H}$ is a separable Hilbert space we shall denote by $\mathscr{L}(\mathscr{H})$, $\mathscr{C}(\mathscr{H})$, $\mathscr{T}(\mathscr{H})$ the Banach spaces of bounded operators with the operator norm, compact operators with the operator norm, and trace class operators with the trace norm, respectively. If $\mathscr{L}_s(\mathscr{H})$, $\mathscr{C}_s(\mathscr{H})$ and $\mathscr{T}_s(\mathscr{H})$ denote the real ordered Banach spaces of self-adjoint operators in the above, then it is well known (see [6]) that we may identify $\mathscr{C}_s(\mathscr{H})^*$ with $\mathscr{T}_s(\mathscr{H})$ and $\mathscr{T}_s(\mathscr{H})^*$ with $\mathscr{L}_s(\mathscr{H})$. Also $(\mathscr{T}_s(\mathscr{H}), \mathrm{tr})$ is a state space in the sense of [4]. If U is a strongly continuous unitary representation of G on $\mathscr{H}$ then the formula

$$sA = UsAUs^*$$

defines a jointly continuous action $G \times \mathscr{C}(\mathscr{H}) \to \mathscr{C}(\mathscr{H})$ and a jointly continuous action $G \times \mathscr{T}(\mathscr{H}) \to \mathscr{T}(\mathscr{H})$ as may be easily verified. However the action $G \times \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$ is only separately continuous and even then only for the weak operator topology of $\mathscr{L}(\mathscr{H})$. The equivariance condition for an instrument $\mathscr{E}$ in this setting is

$$\mathscr{E}(sf, A) = Us[\mathscr{E}(f, Us^*AUs)]\, Us^*$$

and the corresponding condition on its observable $\mathscr{A}_{\mathscr{E}}$ is

$$\mathscr{A}_{\mathscr{E}}(sf) = Us\mathscr{A}_{\mathscr{E}}(f)\, Us^*.$$

We call $\mathscr{A}_{\mathscr{E}}$, U a system of *imprimitivity.*

3. Finite-Dimensional Covariant Transformations

As an introduction we solve a slightly different classification problem which will enable us to find all the covariant instruments which act on finite-dimensional Hilbert spaces.

Let X be a transitive G-space and U, V two strongly continuous unitary representations of G on separable Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ respectively. By a *covariant transformation* $\mathscr{E}$ on X, $\mathscr{H}$, $\mathscr{K}$ we shall mean a bilinear map

$$\mathscr{E} : K(X) \times \mathscr{L}_s(\mathscr{H}) \to \mathscr{T}_s(\mathscr{K})$$

such that

(i) if $f \in K(X^+)$ and $A \in \mathscr{L}_s(\mathscr{H})^+$ then $\mathscr{E}(f, A) \in \mathscr{T}_s(\mathscr{K})^+$;

(ii) if $f \in K(X)$, $A \in \mathscr{L}_s(\mathscr{H})$ and $s \in G$ then

$$\mathscr{E}(sf, A) = Vs\mathscr{E}(f, Us^*AUs)\, Vs^*;$$

(iii) $\mathscr{E}$ is normal; that is if $f \in K(X)^+$ and $A_n \nearrow A$ weakly in $\mathscr{L}_s(\mathscr{H})$ then $\mathscr{E}(f, A_n) \nearrow \mathscr{E}(f, A)$ weakly (or equivalently in the trace norm of $\mathscr{T}_s(\mathscr{K})$).

It is easy to verify that there is a positive linear map $\mathscr{A}_{\mathscr{E}} : K(X) \to \mathscr{T}_s(\mathscr{H})$ defined by

$$\mathrm{tr}[\mathscr{A}_{\mathscr{E}}(f)\, B] = \mathrm{tr}[\mathscr{E}(f, B)]$$

and that for all s in G

$$\mathscr{A}_{\mathscr{E}}(sf) = (Us)\,(\mathscr{A}_{\mathscr{E}} f)\,(Us^*).$$

Now let us write down what will turn out to be the most general covariant transformation. First let H be the subgroup of stability of some point of X; denote by π the natural map $\pi : G \to X$ which allows us to identify X with the H-right coset space of G and let $\theta : K(G) \to K(X)$ be the positive linear map of $K(G)$ onto $K(X)$ defined in [5]. Let U, V be unitary representations of G on the separable Hilbert spaces $\mathscr{H}$, $\mathscr{K}$ respectively; let $T : \mathscr{L}_s(\mathscr{H}) \to \mathscr{T}_s(\mathscr{K})$ be a positive normal linear map and define

$$\mathscr{F} : K(G) \times \mathscr{L}_s(\mathscr{H}) \to \mathscr{T}_s(\mathscr{K})$$

by

$$\mathrm{tr}[\mathscr{F}(f, A)\, B] = \int_G f(s)\, \mathrm{tr}[Vs^* T(UsAUs^*)\, VsB]\, ds$$

where $f \in K(G)$, $A \in \mathscr{L}_s(\mathscr{H})$ and $B \in \mathscr{L}_s(\mathscr{K})$. To see that this is a good definition one notes that as 1 is an order unit in $\mathscr{L}_s(\mathscr{H})$ and $\mathscr{L}_s(\mathscr{K})$ so

$$\mathrm{tr}[Vs^* T(UsAUs^*)\, VsB] \leqslant \| A \|\, \| B \|\, \mathrm{tr}[T(1)]$$

where the left-hand side is a continuous function of s. Also by the Lebesgue dominated convergence theorem the integral represents a normal linear functional of B for each $A \in \mathscr{L}_s(\mathscr{H})^+$ and $f \in K(G)^+$ and so by [7] corresponds to a trace class operator on $\mathscr{K}$. We immediately verify that

$$\begin{aligned}\mathrm{tr}[\mathscr{F}(sf, A)\, B] &= \int_G f(t)\, \mathrm{tr}[VsVt^* T(UtUs^*AUsUt^*)\, VtVs^*B]\, dt \\ &= \mathrm{tr}[\mathscr{F}(f, Us^*AUs)\, Vs^*BVs]\end{aligned}$$

so that

$$\mathscr{F}(sf, A) = Vs\mathscr{F}(f, Us^*AUs)\, Vs^*.$$

Now suppose that for all A in $\mathscr{L}_s(\mathscr{H})$ and h in H

$$T(UhAUh^*) = Vh(TA)\,Vh^*.$$

Then for any s in G

$$\begin{aligned} Vhs^*T(UhsAUhs^*)\,Vhs &= Vs^*Vh^*T(UhUsAUs^*Uh^*)\,VhVs \\ &= Vs^*T(UsAUs^*)\,Vs \end{aligned}$$

so we can define $S : X \times \mathscr{L}_s(\mathscr{H}) \to \mathscr{T}_s(\mathscr{K})$ by

$$S(\pi s, A) = Vs^*T(UsAUs^*)\,Vs.$$

We now note that the functional $f \to \mathrm{tr}[\mathscr{E}(f, I)]$ on $K(X)$ defines a G-invariant positive measure dx on X. We normalize this invariant measure dx on X with respect to the Haar measure ds on G and the Haar measure on H by the equation

$$\int_G f(s)\,ds = \int_X \theta f(x)\,dx$$

where $f \in K(G)$. We define the transformation

$$\mathscr{E} : K(X) \times \mathscr{L}_s(\mathscr{H}) \to \mathscr{T}_s(\mathscr{K})$$

by the equation

$$\mathrm{tr}[\mathscr{E}(f, A)\,B] = \int_X f(x)\,\mathrm{tr}[S(x, A)\,B]\,dx$$

where $f \in K(X)$, $A \in \mathscr{L}_s(\mathscr{H})$ and $B \in \mathscr{L}_s(\mathscr{K})$. If $f = \theta g$ for some g in $K(G)$ then

$$\begin{aligned} \mathrm{tr}[\mathscr{E}(f, A)\,B] &= \int_G g(s)\,\mathrm{tr}[S(\pi s, A)\,B]\,ds \\ &= \int_G g(s)\,\mathrm{tr}[Vs^*T(UsAUs^*)\,VsB]\,ds \\ &= \mathrm{tr}[\mathscr{F}(g, A)]\,B] \end{aligned}$$

and as θ is surjective and satisfies $\theta(sg) = s(\theta g)$ for all s in G and g in $K(G)$ it is immediate that $\mathscr{E}$ is a covariant transformation of the required type. We call this the covariant transformation *induced* by the linear map T.

THEOREM 1. *If $X = G/H$ is a transitive G-space and $\mathscr{E}$ is a covariant transformation on X between the separable Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ then X has an invariant measure and $\mathscr{E}$ is induced by a unique positive normal linear map $T : \mathscr{L}_s(\mathscr{H}) \to \mathscr{T}_s(\mathscr{K})$ satisfying*

$$T(Uh^*AUh) = Vh^*(TA)\, Vh$$

for all h in H.

Proof. Given $\mathscr{E}$ we obtain the invariant dx on X by the formula

$$\int_X f(x)\, dx = \operatorname{tr}[\mathscr{E}(f, 1)]$$

and we suppose the (right) Haar measure of G is normalized with respect to dx as described above. We now pass immediately to the covariant transformation $\mathscr{F} : K(G) \times \mathscr{L}_s(\mathscr{H}) \to \mathscr{T}_s(\mathscr{K})$ defined by

$$\mathscr{F}(f, A) = \mathscr{E}(\theta f, A).$$

If $f \in K(G)^+$ and $A \in \mathscr{L}_s(\mathscr{H})^+$ then

$$\begin{aligned} 0 &\leqslant \operatorname{tr}[\mathscr{F}(f, A)] = \operatorname{tr}[\mathscr{E}(\theta f, A)] \\ &\leqslant \| A \| \int_X (\theta f)(x)\, dx = \| A \| \int_G f(s)\, ds < \infty. \end{aligned}$$

Now let f_n be a sequence in $K(G)^+$ forming an appropriate identity—that is $\int_G f_n(s)\, ds = 1$ for all n and the supports of f_n form a decreasing basic family of neighborhoods of the identity. Let A_n be the set of elements of a countable dense subspace over the rational field of the separable Banach space $\mathscr{C}_s(\mathscr{H})$ and let $A_n = A_{n,1} - A_{n,2}$ where $A_{n,i} \in \mathscr{C}_s(\mathscr{H})^+$ and $\| A_{n,i} \| \leqslant \| A_n \|$. Then as

$$\| \mathscr{F}(f_m, A_{n,i}) \| = \operatorname{tr}[\mathscr{F}(f_m, A_{n,i})] \leqslant \| A_{n,i} \|$$

for all n, i so by the weak compactness of the unit ball of $\mathscr{L}_s(\mathscr{K})$ and a diagonal selection procedure we can find a subsequence of f_m, which we will continue to denote in the same way, and a set of operators $B_{n,i}$ in $\mathscr{L}_s(\mathscr{K})$ such that $\mathscr{F}(f_m, A_{n,i})$ converges weakly to $B_{n,i}$ as $m \to \infty$. It is then immediate that $B_{n,i} \geqslant 0$ and that $B_{n,i}$ are trace class operators with $\operatorname{tr}[B_{n,i}] \leqslant \| A_{n,i} \|$. Defining

$$B_n = B_{n,1} - B_{n,2} \in \mathscr{T}_s(\mathscr{K})$$

it follows that for all n, $\mathscr{F}(f_m, A_n) \to B_n$ weakly as $m \to \infty$ and that $\| B_n \| \leqslant 2 \| A_n \|$. There is a unique linear extension

$$T_1 : \mathscr{C}_s(\mathscr{H}) \to \mathscr{T}_s(\mathscr{K})$$

such that $\| T_1 \| \leqslant 2$ and $T_1 A_n = B_n$ for all n; simple estimates show that for all A in $\mathscr{C}_s(\mathscr{H})$, $\mathscr{F}(f_m, A) \to T_1(A)$ weakly from which it follows that T_1 is a positive linear map. If now $T_1{}^* : \mathscr{C}_s(\mathscr{H}) \to \mathscr{T}_s(\mathscr{H})$ is the adjoint of T_1 and $T : \mathscr{L}_s(\mathscr{H}) \to \mathscr{T}_s(\mathscr{H})$ is the adjoint of $T_1{}^*$ then T is a positive normal linear extension of T_1.

Now that we have defined T we can use it to obtain a new expression for $\mathscr{F}$. If $f \in K(G)$ and $A \in \mathscr{C}_s(\mathscr{H})$ then

$$\mathscr{F}(f, A) = \lim_{m\to\infty} \mathscr{F}(f \circ f_m, A)$$

the limit existing in the norm topology of $\mathscr{T}_s(\mathscr{H})$. For all ξ in $\mathscr{H}$

$$\begin{aligned}\langle \mathscr{F}(f \circ f_m, A)\, \xi, \xi\rangle &= \left\langle \mathscr{F}\left(\int_G f(s)\, (s^{-1} f_m)\, ds, A\right) \xi, \xi \right\rangle \\ &= \int_G f(s) \langle \mathscr{F}(s^{-1} f_m, A)\, \xi, \xi\rangle\, ds \\ &= \int_G f(s) \langle \mathscr{F}(f_m, UsAUs^*)\, Vs\xi, Vs\xi\rangle\, ds\end{aligned}$$

and by the Lebesgue dominated convergence theorem the limit of this as $m \to \infty$ is

$$\int_G f(s) \langle T(UsAUs^*)\, Vs\xi, Vs\xi\rangle\, ds$$

from which it follows that for all f in $K(G)$, A in $\mathscr{C}_s(\mathscr{H})$ and B in $\mathscr{L}_s(\mathscr{H})$

$$\operatorname{tr}[\mathscr{F}(f, A)\, B] = \int_G f(s) \operatorname{tr}[Vs^* T(UsAUs^*)\, VsB]\, ds.$$

The same formula is now valid for all A in $\mathscr{L}_s(\mathscr{H})$ because of the normality of T.

If $h \in H$ and $g \in K(G)$ is defined by $g(s) = \delta_H(h) f(hs)$ then $\theta f = \theta g$, so $\mathscr{F}(f, A) = \mathscr{F}(g, A)$ for all A in $\mathscr{C}_s(\mathscr{H})$. Therefore for all A in $\mathscr{C}_s(\mathscr{H})$ and B in $\mathscr{L}_s(\mathscr{H})$

$$\begin{aligned}&\int_G f(s) \operatorname{tr}[Vs^* T(UsAUs^*)\, VsB]\, ds \\ &= \operatorname{tr}[\mathscr{F}(f, A)\, B] = \operatorname{tr}[\mathscr{F}(g, A)\, B] \\ &= \int_G \delta_H(h) f(hs) \operatorname{tr}[Vs^* T(UsAUs^*)\, VsB]\, ds \\ &= \int_G f(s) \operatorname{tr}[Vs^* Vh T(Uh^* UsAUs^* Uh)\, Vh^* VsB]\, ds\end{aligned}$$

Observing that the integrands are continuous functions of s and letting f run through an approximate identity in $K(G)$ it follows that

$$\operatorname{tr}[T(A)\,B] = \operatorname{tr}[VhT(Uh^*AUh)\,Vh^*B]$$

so that

$$Vh^*(TA)\,Vh = T(Uh^*AUh).$$

Since T is normal this formula now holds for all A in $\mathscr{L}_s(\mathscr{H})$ as well.

This completes the proof of the existence half of the theorem and the uniqueness follows from the fact that if f_m is an approximate identity in $K(G)$, $A \in \mathscr{C}_s(\mathscr{H})$ and $B \in \mathscr{L}_s(\mathscr{K})$ then

$$\operatorname{tr}[T(A)\,B] = \lim_{m\to\infty} \operatorname{tr}[\mathscr{E}(\theta f_m, A)\,B].$$

We now make some comments on the application of this theorem to finite-dimensional instruments. If $\mathscr{H} = \mathscr{K}$ is an n-dimensional Hilbert space and $U = V$ then a covariant transformation $\mathscr{E}$ on X is an instrument if and only if for any increasing sequence f_n in $K(X)^+$ which converges pointwise to the identity and any A in $\mathscr{L}(\mathscr{H})^+$

$$\operatorname{tr}[\mathscr{E}(f_n, A)] \to \operatorname{tr}[A].$$

Applying this when $A = 1$ we see that $\int_X f_n(x)\,dx \to n$ so dx is a bounded invariant measure on X. It is now reasonable to consider only the case where G is compact, since then any transitive G-space X is compact and has a bounded invariant measure, while G itself has plenty of finite-dimensional representations. Letting all the invariant measures be normalized to have unit total mass we obtain

Corollary 2. *If U is a representation of the compact group G on the n-dimensional Hilbert space $\mathscr{H}$ and $X = G/H$ is a transitive G-space then the formula*

$$\mathscr{E}(\theta f, A) = \int_G f(s)\,Us^*T(UsAUs)\,Us\,ds$$

where $f \in K(G)$ and $A \in \mathscr{L}_s(\mathscr{H})$, sets up a one-one correspondence between the covariant instruments $\mathscr{E}$ on X, $\mathscr{H}$ and the positive linear maps $T : \mathscr{L}_s(\mathscr{H}) \to \mathscr{L}_s(\mathscr{H})$ satisfying

(i) *for all h in H and A in $\mathscr{L}_s(\mathscr{H})$*

$$T(UhAUh^*) = Uh(TA)\,Uh^*;$$

(ii) *if* $A \in \mathscr{L}_s(\mathscr{H})$ *satisfies* $AUs = UsA$ *for all* s *in* G, $\mathrm{tr}[A] = \mathrm{tr}[TA]$.

If X *is infinite then the observable of such an instrument cannot be a projection-valued measure.*

Remarks. As H is compact so the positive maps $T : \mathscr{L}_s(\mathscr{H}) \to \mathscr{L}_s(\mathscr{H})$ satisfying condition (i) are precisely those of the form

$$T(A) = \int_H Uh^* S(UhAUh^*)\, Uh\, dh$$

where $S : \mathscr{L}_s(\mathscr{H}) \to \mathscr{L}_s(\mathscr{H})$ is an arbitrary positive linear map. If U is irreducible condition (ii) reduces to $\mathrm{tr}[T(1)] = n$.

Proof. The condition that

$$\mathrm{tr}[\mathscr{E}(1, A)] = \mathrm{tr}[A]$$

for all A in $\mathscr{L}_s(\mathscr{H})$ can be rewritten as

$$\mathrm{tr}[A] = \int_G \mathrm{tr}[T(UsAUs^*)]\, ds$$

or

$$\mathrm{tr}\left[\int_G UsAUs^*\, ds\right] = \mathrm{tr}\left[T\left(\int_G UsAUs^*\, ds\right)\right]$$

and the operators of the form $B = \int_G UsAUs^*\, ds$ are precisely those satisfying condition (ii). If $\mathscr{A}_\mathscr{E}$ is a projection-valued measure then $\mathscr{A}_\mathscr{E}$, U is a projection system of imprimitivity of X, G and by [8, 9] if $\mathscr{H}$ is finite-dimensional then X is finite.

4. Instruments whose Observables are Projection-Valued Measures

Since it has been generally accepted that there is a one-one correspondence between observables and projection-valued measures (see [10, 11]) we turn now to the classification of those covariant instruments $\mathscr{E}$ on a Borel space X such that $\mathscr{E}^*(E, I)$ is a projection for all $E \subseteq X$.

We first review some of the theory of induced representations (see [5, 8, 9]) and for the sake of technical simplicity restrict ourselves to the case where G is a separable locally compact group and H is a compact sub-group of G.

If ρ is a strongly continuous unitary representation of H on a separable Hilbert space $\mathscr{K}$ then we denote by $\mathscr{L}_H^2\{G, \mathscr{K}\}$ the Hilbert space of Borel functions $f : G \to \mathscr{K}$ such that $\int_G \|f(s)\|^2 \, ds < \infty$ and for all s in G and h in H, $f(hs) = \rho(h) f(s)$, two functions being identified if they are equal almost everywhere. The induced representation U of G on $\mathscr{H} = \mathscr{L}_H^2\{G, \mathscr{K}\}$ is defined by the formula

$$(Utf)(s) = f(st)$$

and the projection-valued measure P on X with values in $\mathscr{L}(\mathscr{H})$ is defined by

$$\{(Pg)f\}(s) = g(\pi s) f(s)$$

for all g in $B(X)$ and f in $\mathscr{H}$. Now let $\mathscr{L}_H^\infty\{G, \mathscr{L}(\mathscr{K})\}$ denote the *-algebra of bounded Borel function f on G to $\mathscr{L}(\mathscr{K})$ such that for all s in G and h in H

$$f(hs) = \rho(h) f(s)\, \rho(h)^*$$

two functions being identified if they are equal almost everywhere. If $f \in \mathscr{L}_H^2\{G, \mathscr{K}\}$ and $g \in \mathscr{L}_H^\infty\{G, \mathscr{L}(\mathscr{K})\}$ the formula

$$\{(\lambda_1 g) f\}(s) = f(s)\, g(s)$$

defines a *-isomorphism of $\mathscr{L}_H^\infty\{G, \mathscr{L}(\mathscr{K})\}$ into $\mathscr{L}(\mathscr{H})$ and the image is precisely the commutant in $\mathscr{L}(\mathscr{H})$ of the commutative von Neumann algebra $P\{B(X)\}$ (see [8]). Finally we define $\mathscr{L}_H^1\{G, \mathscr{T}(\mathscr{K})\}$ as the Banach space of all Borel functions f from G to $\mathscr{T}(\mathscr{K})$ such that $\int_G \|f(s)\| \, ds < \infty$ and for all s in G and h in H

$$f(hs) = \rho(h) f(s) \rho(h)^*$$

two functions again being identified if they are equal almost everywhere. The dual space of $\mathscr{L}_H^1\{G, \mathscr{T}(\mathscr{K})\}$ is $\mathscr{L}_H^\infty\{G, \mathscr{L}(\mathscr{K})\}$ by [12], and the adjoint of $\lambda_1 : \mathscr{L}_H^\infty\{G, \mathscr{L}(\mathscr{K})\} \to \mathscr{L}(\mathscr{H})$ is a map $\lambda : \mathscr{T}(\mathscr{H}) \to \mathscr{L}_H^1\{G, \mathscr{T}(\mathscr{K})\}$. It is immediate that if $f \in \mathscr{L}_H^2\{G, \mathscr{K}\}$ then

$$\{\lambda(f \otimes \bar{f})\}(s) = f(s) \otimes \overline{f(s)}$$

and that for all t in G and A in $\mathscr{T}(\mathscr{H})$

$$\{\lambda(UtAUt^*)\}(s) = (\lambda A)(st)$$

for almost all s in G.

Now let $T : \mathcal{T}_s(\mathcal{K}) \to \mathcal{T}_s(\mathcal{H})$ be a positive linear map such that for all A in $\mathcal{T}_s(\mathcal{K})$

$$\mathrm{tr}[TA] = \mathrm{tr}[A]$$

and define $\mathcal{E} : K(X) \times \mathcal{T}_s(\mathcal{H}) \to \mathcal{T}_s(\mathcal{H})$ by

$$\mathcal{E}(f, A) = \int_G f(\pi s)\, Us^* T\{(\lambda A)(s)\}\, Us\, ds$$

the integral converging in the weak topology of $\mathcal{T}_s(\mathcal{H})$. Then for all t in G, f in $K(X)$ and A in $\mathcal{T}_s(\mathcal{H})$

$$\begin{aligned}\mathcal{E}(tf, A) &= \int_G f(\pi s)\, Us^* t^{-1} T\{(\lambda A)(st^{-1})\}\, Ust^{-1}\, ds \\ &= Ut \int_G f(\pi s)\, Us^* T\{\lambda(Ut^* A Ut)(s)\}\, Us\, ds\, Ut^* \\ &= Ut\mathcal{E}(f, Ut^* A Ut)\, Ut^*\end{aligned}$$

while

$$\begin{aligned}\mathrm{tr}[\mathcal{E}(f, A)] &= \int_G f(\pi s)\, \mathrm{tr}[(\lambda A)(s)]\, ds \\ &= \mathrm{tr}[\lambda_1(f1)\, A] \\ &= \mathrm{tr}[P(f)\, A]\end{aligned}$$

so $\mathcal{E}$ is a covariant instrument whose observable is a projection-valued measure. We call this the instrument *induced* by the representation ρ of H and the map $T : \mathcal{T}_s(\mathcal{K}) \to \mathcal{T}_s(\mathcal{H})$.

THEOREM 3. *Let H be a compact subgroup of the separable locally compact group G and let X be the H-right coset space of G. If $\mathcal{E}$, U is a covariant instrument on X, G acting on a separable Hilbert space $\mathcal{H}$ such that $\mathcal{A}_g$, U is a projective system of imprimitivity on X, G induced by a representation ρ of H on a Hilbert space $\mathcal{K}$, then $\mathcal{E}$ is induced by a unique positive linear map $T : \mathcal{T}_s(\mathcal{K}) \to \mathcal{T}_s(\mathcal{H})$ satisfying*

(i) $\mathrm{tr}[TA] = \mathrm{tr}[A]$ *for all A in $\mathcal{T}_s(\mathcal{K})$;*

(ii) $T((\rho h)\, A(\rho h)^*) = Uh(TA)\, Uh^*$ *for all A in $\mathcal{T}_s(\mathcal{K})$ and all h in H.*

Remark. If H is not compact there is an analogous result which is slightly more complicated to formulate since $\mathcal{L}_H^1\{G, \mathcal{T}(\mathcal{K})\}$ must be replaced by a space whose definition is somewhat more difficult.

However our theorem suffices for the case in which we are most interested, where $G = \mathbf{R}^n \circledS SO(n)$ and $H = SO(n)$.

Proof. As $\mathscr{A}_{\mathscr{E}}$, U is a projective system of imprimitivity so by Mackey's imprimitivity theorem there is a separable Hilbert space $\mathscr{K}$ and a representation ρ of H on $\mathscr{K}$ such that $\mathscr{H} \simeq \mathscr{L}_H^2\{G, \mathscr{K}\}$ under an isomorphism which identifies U with the induced representation and $\mathscr{A}_{\mathscr{E}}$ with P. Associated with $\mathscr{E}$ there is a positive bilinear map $\mathscr{E}^* : B(X) \times \mathscr{L}_s(\mathscr{H}) \to \mathscr{L}_s(\mathscr{H})$ defined by

$$\operatorname{tr}[\mathscr{E}^*(f, A)\, B] = \operatorname{tr}[A\mathscr{E}(f, B)]$$

where $f \in B(X)$, $A \in \mathscr{L}_s(\mathscr{H})$ and $B \in \mathscr{T}_s(\mathscr{H})$. For all B in $\mathscr{L}_s(\mathscr{H})$ and Borel sets $E \subseteq X$

$$\operatorname{tr}[\mathscr{E}^*(\chi_E, 1)\, B] = \operatorname{tr}[\mathscr{E}(\chi_E, B)] = \operatorname{tr}[P_E B]$$

so $\mathscr{E}^*(\chi_E, 1) = P_E$. For any E, F in X and B in $\mathscr{L}_s(\mathscr{H})^+$

$$0 \leqslant \mathscr{E}^*(\chi_{E\cap F}, B) \leqslant \| B \|\, \mathscr{E}^*(\chi_{E\cap F}, 1) = \| B \|\, P_{E\cap F}$$

from which it is clear that

$$\mathscr{E}^*(\chi_{E\cap F}, B)\, P_F = P_F \mathscr{E}^*(\chi_{E\cap F}, B) = \mathscr{E}^*(\chi_{E\cap F}, B).$$

This and the similar equation

$$\mathscr{E}^*(\chi_{E\cap(X-F)}, B)\, P_F = P_F \mathscr{E}^*(\chi_{E\cap(X-F)}, B) = 0$$

give on addition

$$\mathscr{E}^*(\chi_E, B)\, P_F = P_F \mathscr{E}^*(\chi_E, B) = \mathscr{E}^*(\chi_{E\cap F}, B)$$

from which it follows that for all f, g in $B(X)$ and B in $\mathscr{L}_s(\mathscr{H})$, $P(g)$ commutes with $\mathscr{E}^*(f, B)$.

We now know that $\mathscr{E}^* = \lambda \mathscr{E}_1^*$ where

$$\mathscr{E}_1^* : B(X) \times \mathscr{L}_s(\mathscr{H}) \to \mathscr{L}_H^\infty(G, \mathscr{L}(\mathscr{K})\}$$

and want to prove the much stronger result that for all f in $C_\infty(X)$ and A in $\mathscr{C}_s(\mathscr{H})$, $\mathscr{E}_1^*(f, A) \in C_H\{G, \mathscr{L}(\mathscr{K})\}$, where $C_\infty(X)$ is the Banach space of real continuous functions on X which vanish at infinity with the sup norm, and $C_H\{G, \mathscr{L}(\mathscr{K})\}$ is the space of continuous functions in $\mathscr{L}_H^\infty\{G, \mathscr{L}(\mathscr{H})\}$. First note that $\mathscr{E}_1{}^*$ is bilinear and for any f in $C_\infty(X)$ and A in $\mathscr{C}_s(\mathscr{H})$

$$\| \mathscr{E}_1{}^*(f, A)\| \leqslant \|f\| \, \| A \|$$

so there exists a linear map

$$\mathscr{E}_2{}^* : C_\infty(X) \mathbin{\widehat{\otimes}} \mathscr{C}_s(\mathscr{H}) \to \mathscr{L}_H^\infty\{G, \mathscr{L}(\mathscr{H})\}$$

such that $\| \mathscr{E}_2{}^* \| \leqslant 1$ and $\mathscr{E}_2{}^*(f \otimes A) = \mathscr{E}_1{}^*(f, A)$, where $\widehat{\otimes}$ denotes the projective tensor product, as defined in [13]. As $C_\infty(X) \mathbin{\widehat{\otimes}} \mathscr{C}_s(\mathscr{H})$ is separable, by [12] we can lift $\mathscr{E}_2{}^*$ to a linear map

$$\mathscr{E}_3{}^* : C_\infty(X) \mathbin{\widehat{\otimes}} \mathscr{C}_s(\mathscr{H}) \to B_H\{G, \mathscr{L}(\mathscr{H})\}$$

such that $\| \mathscr{E}_3{}^* \| \leqslant 1$ and $\mathscr{E}_2{}^* = \mu \mathscr{E}_3{}^*$ where

$$\mu : B_H\{G, \mathscr{L}(\mathscr{H})\} \to \mathscr{L}_H^\infty\{G, \mathscr{L}(\mathscr{H})\}$$

is the map identifying two functions equal almost everywhere. We now fix A in $\mathscr{C}_s(\mathscr{H})$ and f in $K(X)$ and define

$$Y : G \times G \times \quad \to \mathscr{L}_s(\mathscr{H})$$

by

$$Y(r, s, t) = \mathscr{E}_3{}^*\{(rf) \otimes (UsAUs^*)\}\,(t).$$

For each r, s this is a Borel function of t and for each t it is a continuous function of r, s; therefore Y is a Borel function of r, s, t jointly. Now for each r, s in G

$$\mathscr{E}^*(rsf, UrAUr^*) = Ur\mathscr{E}^*(sf, A)\, Ur^*$$

and so

$$Y(rs, r, t) = Y(s, e, tr)$$

for almost all t in G. Therefore these two functions of r, s, t are equal almost everywhere on $G \times G \times G$ and for at least one a in G {in fact for almost all a in G}

$$Y(rs, r, a) = Y(s, e, ar)$$

for almost all r, s in $G \times G$. Now for any g in $K(G)^+$, A in $\mathscr{C}_s(\mathscr{H})^+$ and B in $\mathscr{L}^\infty\{G, \mathscr{L}(\mathscr{H})\}$

$$\int_G \operatorname{tr}[\mathscr{E}_3^*\{g \circ f) \otimes A\}(r)\, B(r)]\, dr$$

$$= \int_G \int_G g(s) \operatorname{tr}[\mathscr{E}_3^*\{(s^{-1}f) \otimes A\}(r)\, B(r)]\, ds\, dr$$

$$= \delta_G(a) \int_G \int_G g(s) \operatorname{tr}[\mathscr{E}_3^*\{(s^{-1}f) \otimes A\}(ar)\, B(ar)]\, ds\, dr$$

$$= \delta_G(a) \int_G \int_G g(s) \operatorname{tr}[Y(s^{-1}, e, ar)\, B(ar)]\, ds\, dr$$

$$= \delta_G(a) \int_G \int_G g(s) \operatorname{tr}[Y(rs^{-1}, r, a)\, B(ar)]\, dr\, ds$$

$$= \int_G \int_G g(s) \operatorname{tr}[\mathscr{E}_3^*\{(a^{-1}rs^{-1}f) \otimes (Ua^{-1}rAUa^{-1}r^*)\}(a)\, B(r)]\, dr\, ds$$

Therefore

$$\mathscr{E}_3^*\{(g \circ f) \otimes A\}(r) = \int_G g(s)\, \mathscr{E}_3^*\{(a^{-1}rs^{-1}f) \otimes (Ua^{-1}rAUa^{-1}r^*)\}(a)\, ds$$

for almost all r in G, and the integral can be seen to define a continuous function of r.

As functions of the form $g \circ f$ where $g \in K(G)$ and $f \in K(X)$ are uniformly dense in $C_\infty(X)$ so by taking limits and using the fact that $\| \mathscr{E}_3^* \| \leqslant 1$ it follows that $\mathscr{E}_3^*(f, A)$ is equal almost everywhere to a continuous function for all f in $C_\infty(X)$ and all A in $\mathscr{C}_s(\mathscr{H})$. We denote this continuous function, which is necessarily unique, by $\mathscr{E}_4^*(f, A)$ and see that $\mathscr{E}_4^* : C_\infty(S) \times \mathscr{C}_s(\mathscr{H}) \to C_H\{G, \mathscr{L}(\mathscr{H})\}$ has the following properties:

(i) $\mathscr{E}_4^*$ is positive and bilinear with

$$\| \mathscr{E}_4^*(f, A)(r) \| \leqslant \|f\|\, \|A\|$$

for all f in $C_\infty(X)$, A in $\mathscr{C}_s(\mathscr{H})$ and r in G;

(ii) $\mathscr{E}_4^*(f, A)(hr) = \rho(h)\, [\mathscr{E}_4^*(f, A)(r)]\, \rho(h)^*$ for all f in $C_\infty(X)$, A in $\mathscr{C}_s(\mathscr{H})$, h in H and r in G;

(iii) $\mathscr{E}_4^*(sf, A)(t) = \mathscr{E}_4^*(f, Us^*AUs)(ts)$ for all f in $C_\infty(X)$, A in $\mathscr{C}_s(\mathscr{H})$ and s, t in G.

Now let $f \in K(X)$ satisfy $f(\pi e) = 0$ and let g_n be a decreasing sequence in $K(X)^+$ such that $0 \leqslant g_n \leqslant 1$ and both

$$V_n = \{x \in X : g_n(x) = 1\} \qquad \text{and} \qquad W_n = \{x \in X : g_n(x) \neq 0\}$$

form decreasing basic families of neighborhoods of πe in X. Then

$$-\chi_{X-V_n} \| f \| \leqslant f(1 - g_n) \leqslant \chi_{X-V_n} \| f \|$$

so

$$-\| f \| \| A \| P_{X-V_n} \leqslant \mathscr{E}^*(f - fg_n, A) \leqslant \| f \| \| A \| P_{X-V_n}$$

Therefore

$$\mathscr{E}_4^*(f - fg_n, A)(e) = 0$$

so

$$\mathscr{E}_4^*(f, A)(e) = \mathscr{E}_4^*(fg_n, A)(e)$$

But

$$\| \mathscr{E}^*(fg_n, A) \| \leqslant \| A \| \| fg_n \| \to 0$$

so

$$\mathscr{E}_4^*(f, A)(e) = 0.$$

It now follows that $\mathscr{E}_4^*(f, A)(e)$ depends only on $f(\pi e)$ and A. That is there exists a positive linear map $T_1 : \mathscr{C}_s(\mathscr{H}) \to \mathscr{L}_s(\mathscr{H})$ with $\| T_1 \| \leqslant 1$ such that for all f in $K(X)$ and A in $\mathscr{C}_s(\mathscr{H})$

$$\mathscr{E}_4^*(f, A)(e) = T_1(A) f(\pi e).$$

By conditions (i), (ii) and (iii) above we see that for all A in $\mathscr{C}_s(\mathscr{H})$ and all h in H

$$T_1(UhAUh^*) = \rho(h)\, T_1(A)\, \rho(h)^*$$

and that $\mathscr{E}_4^*$ is determined from T_1 by the equation

$$\mathscr{E}_4^*(f, A)(s) = T_1(UsAUs^*) f(\pi s).$$

The adjoint T of T_1 is a positive linear map $T : \mathscr{T}_s(\mathscr{H}) \to \mathscr{T}_s(\mathscr{H})$ satisfying $\| T \| \leqslant 1$ and

$$T(\rho(h)\, A \rho(h)^*) = Uh(TA)\, Uh^*$$

for all A in $\mathcal{T}_s(\mathcal{K})$ and all h in H. Also for all A in $\mathcal{T}_s(\mathcal{H})$, B in $\mathcal{C}_s(\mathcal{H})$ and f in $K(X)$

$$\begin{aligned}\operatorname{tr}[\mathscr{E}(f, A)\, B] &= \operatorname{tr}[A\mathscr{E}^*(f, B)] \\ &= \int_G \operatorname{tr}[(\lambda A)\,(s)\,\mathscr{E}_4{}^*(f, B)\,(s)]\, ds \\ &= \int_G f(\pi s) \operatorname{tr}[(\lambda A)\,(s)\, T_1(UsBUs^*)]\, ds\end{aligned}$$

Therefore

$$\operatorname{tr}[\mathscr{E}(f, A)\, B] = \int_G f(\pi s) \operatorname{tr}[Us^* T\{(\lambda A)\,(s)\}\, UsB]\, ds$$

and by monotone convergence arguments this formula now also holds for all B in $\mathcal{L}_s(\mathcal{H})$.

To complete the existence part of the proof, we have now only to show that for all A in $\mathcal{T}_s(\mathcal{K})$, $\operatorname{tr}[T(A)] = \operatorname{tr}[A]$, and it is clearly enough to do this in the particular case where $A = \alpha \otimes \bar{\alpha}$ for some α in $\mathcal{K}$. Given g in $K(G)$ the function $g^\alpha : G \to \mathcal{K}$ defined by

$$g^\alpha(s) = \int_H g(hs)\, \rho(h)^*\, \alpha\, dh$$

is a continuous function in $\mathcal{L}_H{}^2\{G, \mathcal{K}\}$ and so if $f \in K(X)$

$$\operatorname{tr}[\mathscr{E}(f, g^\alpha \otimes g^{\bar{\alpha}})] = \int_G f(\pi s) \operatorname{tr}[T(g^\alpha(s) \otimes \overline{g^\alpha(s)})]\, ds.$$

Letting f_n be an increasing sequence in $K(X)^+$ converging pointwise to the identity we obtain

$$\begin{aligned}\| g^\alpha \|^2 = \operatorname{tr}[g^\alpha \otimes \bar{g}^\alpha] &= \int_G \operatorname{tr}\{T[g^\alpha(s) \otimes \overline{g^\alpha(s)}]\}\, ds \\ &= \int_G \| T(g^\alpha(s) \otimes \overline{g^\alpha(s)})\|\, ds \\ &\leqslant \int_G \| g^\alpha(s) \otimes \overline{g^\alpha(s)}\|\, ds = \| g^\alpha \|^2\end{aligned}$$

and as all the functions involved are continuous

$$\operatorname{tr}\{T[g^\alpha(e) \otimes \overline{g^\alpha(e)}]\} = \| g^\alpha(e)\|^2.$$

Letting g_m run through an approximate identity in $K(G)$ gives by continuity arguments

$$\mathrm{tr}[T(\alpha \otimes \bar{\alpha})] = \| \alpha \|^2 = \mathrm{tr}[\alpha \otimes \bar{\alpha}].$$

The final part of the theorem, that T is unique, comes from the formula

$$\mathrm{tr}[T(\alpha \otimes \bar{\alpha}) B] = \lim_{m\to\infty} \lim_{n\to\infty} \mathrm{tr}[\mathscr{E}(f_n, g_m{}^{\alpha} \otimes \overline{g_m{}^{\alpha}}) B]$$

where $B \in \mathscr{L}_s(\mathscr{H})$ and α, f_n, g_n are as defined above. We leave the reader to verify this himself.

5. Generalized Position-Measuring Instruments

We now want to construct a class of covariant instruments $\mathscr{E}$ such that the observable $A(E) = \mathscr{E}^*(E, I)$ satisfies $A(E)\,A(F) = A(F)\,A(E)$ for all $E, F \subseteq X$. We give up the assumption that each operator $A(E)$ is a projection.

Since it is not at all clear in what sense an operator-valued probability measure might be regarded as giving one a measure of position we shall discuss this point first, and have no difficulty in formulating the ideas for a general state space (V, τ). Given two observables $\mathscr{A}$ on X and $\mathscr{B}$ on Y with values in V^* we say that $\mathscr{A} \leqslant \mathscr{B}$, in words $\mathscr{A}$ *gives less information than* $\mathscr{B}$, if for any two states A, B in V^+ such that $\langle \mathscr{B}(F), A\rangle = \langle \mathscr{B}(F), B\rangle$ for all Borel sets $F \subseteq Y$ it follows that $\langle \mathscr{A}(E), A\rangle = \langle \mathscr{A}(E), B\rangle$ for all Borel sets $E \subseteq X$. If $L_{\mathscr{B}} \subseteq V^*$ is the weak*-closed linear subspace of V^* generated by $\{\mathscr{B}(F) : F \subseteq Y\}$ then $\mathscr{A} \leqslant \mathscr{B}$ if and only if $L_{\mathscr{A}} \subseteq L_{\mathscr{B}}$. In particular we say that two observables $\mathscr{A}$ and $\mathscr{B}$ are *equivalent* or *give the same information* if $\mathscr{A} \leqslant \mathscr{B}$ and $\mathscr{B} \leqslant \mathscr{A}$, or if $L_{\mathscr{A}} = L_{\mathscr{B}}$.

In the case where $V = \mathscr{T}_s(\mathscr{H})$ and $\mathscr{B}$ is a projection-valued position (momentum) observable we say that $\mathscr{A}$ is a *modified position* (*resp. momentum*) *observable* if $\mathscr{A} \leqslant \mathscr{B}$. Here $L_{\mathscr{B}}$ is a commutative von Neumann algebra in $\mathscr{L}(\mathscr{H})$ so any modified position observable $\mathscr{A}$ is a commutative operator-valued measure in the sense that

$$A(E)\,A(F) = A(F)\,A(E) \qquad \text{for all} \qquad E, F \subseteq X.$$

For our purposes it will not be necessary to examine a very general system and we restrict ourselves to the case where G is a semidirect product. If X is a separable locally compact abelian group with

Haar measure dx and H is a compact separable group acting on the right as a group of automorphisms of X, it is known that H leaves the Haar measure dx of X invariant. We define $G = H \circledS X$ as the semi-direct product, giving it the product topology and the multiplication

$$(h, x)(h', x') = (hh', x \cdot h' + x').$$

X is transitive G-space under the action

$$x(h', x') = x \cdot h' + x'$$

and the subgroup of stability of $e \in X$ is H while dx is a G-invariant measure for X. If ρ is a representation of H on the separable Hilbert space $\mathscr{K}$ then there is a unitary isomorphism φ of $\mathscr{L}^2\{X, K\}$ onto $\mathscr{L}_H{}^2\{G, \mathscr{K}\} = \mathscr{H}$ given by

$$(\varphi f)(h, x) = \rho(h) f(x).$$

Regarding the induced system of imprimitivity P, U of X, G as acting on $\mathscr{L}^2\{X, \mathscr{K}\}$ it is easy to verify that for all f in $\mathscr{L}^2\{X, \mathscr{K}\}$, g in $K(X)$ and (h, y) in G

$$\{(Pg)f\}(x) = g(x) f(x)$$
$$\{U(h, Y)f\}(x) = \rho(h) f(x \cdot h + y)$$

these equations holding for almost all x in X.

THEOREM 4. *Let $G = H \circledS X$ be the semi-direct product of the separable locally compact abelian group X by the separable compact group H and let U, P be the system of imprimitivity of G, X induced on the separable Hilbert space $\mathscr{H}$ by a given representation ρ of H. Let α be a complex-valued bounded Borel function on X such that $\alpha(x \cdot h) = \alpha(x)$ for all x in X and h in H and $\int_X |\alpha(x)|^2 \, dx = 1$. Then the map*

$$\mathscr{E}_\alpha : K(X) \times \mathscr{T}_s(\mathscr{H}) \to \mathscr{T}_s(\mathscr{H})$$

defined by the equation

$$\operatorname{tr}[\mathscr{E}_\alpha(f, A) B] = \int_X f(x) \operatorname{tr}[Ux^* P(\bar{\alpha})\, UxAUx^* P(\alpha)\, UxB]\, dx$$

is a covariant instrument with respect to U *and its observable* $\mathscr{A}_{\mathscr{E}_\alpha} : K(X) \to \mathscr{L}_s(\mathscr{H})$ *is given by*

$$\mathrm{tr}[\mathscr{A}_{\mathscr{E}_\alpha}(f)\, A] = \mathrm{tr}[P(f \circ |\, \alpha\, |^2)\, A]$$

and so is a modified P-observable.

Proof. We first observe that for all f in $K(X)^+$ and A in $\mathscr{T}_s(\mathscr{H})^+$ the above integral represents a normal positive linear functional of $B \in \mathscr{L}_s(\mathscr{H})$ and so $\mathscr{E}_\alpha(f, A) \in \mathscr{T}_s(\mathscr{H})^+$. We now show that $\mathscr{E}_\alpha$ has the right covariance properties with respect to G. For any y in X

$$\begin{aligned}\mathrm{tr}[\mathscr{E}_\alpha(yf, A)\, B] &= \int_X f(x+y)\, \mathrm{tr}[Ux^*P(\bar{\alpha})\, UxAUx^*P(\alpha)\, UxB]\, dx \\ &= \int_X f(x)\, \mathrm{tr}[UyUx^*P(\bar{\alpha})\, UxUy^*AUyUx^*P(\alpha)\, UxUy^*B]\, dx \\ &= \mathrm{tr}[\mathscr{E}_\alpha(f, Uy^*AUy)\, (Uy^*BUy)]\end{aligned}$$

so

$$\mathscr{E}_\alpha(yf, A) = Uy\mathscr{E}_\alpha(f, Uy^*AUy)\, Uy^*.$$

Similarly for any h in H

$$\begin{aligned}\mathrm{tr}[\mathscr{E}_\alpha(hf, A)\, B] &= \int_X f(x)\, \mathrm{tr}[Ux \cdot h^{-1*}P(\bar{\alpha})\, Ux \cdot h^{-1}AUx \\ &\qquad \cdot h^{-1*}P(\alpha)\, Ux \cdot h^{-1}B]\, dx \\ &= \int_X f(x)\, \mathrm{tr}[UhUx^*Uh^*P(\bar{\alpha})\, UhUxUh^*AUhUx^*Uh^*P(\alpha) \\ &\qquad \times UhUxUh^*B]\, dx \\ &= \int_X f(x)\, \mathrm{tr}[UhUx^*P(\bar{\alpha})\, UxUh^*AUhUx^*P(\alpha)\, UxUh^*B]\, dx \\ &= \mathrm{tr}[\mathscr{E}_\alpha(f, Uh^*AUh)\, (Uh^*BUh)]\end{aligned}$$

so

$$\mathscr{E}_\alpha(hf, A) = Uh\mathscr{E}_\alpha(f, Uh^*AUh)\, Uh^*.$$

The same equation now holds for all s in G because every element of

G is a product of elements of H and X. The observable $\mathscr{A}_{\mathscr{E}_\alpha}$ of $\mathscr{E}_\alpha$ is given by

$$\begin{aligned}
\operatorname{tr}[\mathscr{A}_{\mathscr{E}_\alpha}(f)\,A] &= \operatorname{tr}[\mathscr{E}_\alpha(f, A)] \\
&= \int_X f(x)\operatorname{tr}[Ux^*P(\bar{\alpha})\,UxAUx^*P(\alpha)\,Ux]\,dx \\
&= \int_X f(x)\operatorname{tr}[Ux^*P(|\,\alpha\,|^2)\,UxA]\,dx \\
&= \int_X f(x)\operatorname{tr}[P(x^{-1}\,|\,\alpha\,|^2)\,A]\,dx \\
&= \operatorname{tr}\left[P\left(\int_X f(x)\,(x^{-1}\,|\,\alpha\,|^2)\,dx\right)A\right] \\
&= \operatorname{tr}[P(f\circ|\,\alpha\,|^2)\,A].
\end{aligned}$$

We can now complete the proof that $\mathscr{E}_\alpha$ is an instrument. If f_n is an increasing sequence in $K(X)^+$ converging pointwise to the identity then $f_n \circ |\,\alpha\,|^2$ converges monotonically to the identity so

$$\operatorname{tr}[\mathscr{E}_\alpha(f_n, A)] \to \operatorname{tr}[P(1)\,A] = \operatorname{tr}[A].$$

As the observable of $\mathscr{E}_\alpha$ is given by $\mathscr{A}_{\mathscr{E}_\alpha}(E) = P(\chi_E \circ |\,\alpha\,|^2)$ it is immediate that $\mathscr{A}_{\mathscr{E}_\alpha}$, U is a commutative system of imprimitivity of X, G. If P is the position observable of some system then $\mathscr{A}_{\mathscr{E}_\alpha}$ is a modified position observable as defined above.

It is interesting to inquire under what conditions P and $\mathscr{A}_{\mathscr{E}_\alpha}$ give the same information. For any A in $\mathscr{T}_s(\mathscr{H})$ there exists φ in $\mathscr{L}^1(X)$ such that for all f in $C_\infty(X)$

$$\operatorname{tr}[P(f)\,A] = \int_X f(x)\,\varphi(x)\,dx.$$

Now if $\operatorname{tr}[\mathscr{A}_{\mathscr{E}_\alpha}(f)] = 0$ for all f in $C_\infty(X)$ then in particular putting $f = g^\wedge$ where $g \in K(\hat{X})$

$$\begin{aligned}
0 &= \int_X (f\circ|\,\alpha\,|^2)\,(x)\,\varphi(x)\,dx \\
&= \int_X\int_X \hat{g}(y)\,|\,\alpha\,|^2\,(x-y)\,\varphi(x)\,dx\,dy \\
&= \int_{\hat{X}} g(z)\,|\,\alpha\,|^{2*\wedge}\,(z)\,\hat{\varphi}(z)\,dz
\end{aligned}$$

so $|\alpha|^{2\wedge}(z)\,\hat{\varphi}(z) = 0$ for all z in $\hat{X}$. If $|\alpha|^{2\wedge}$ is non-zero on an open dense set in $\hat{X}$ then $\hat{\varphi} = 0$ so $\varphi = 0$. Therefore P and $\mathscr{A}_{\mathscr{E}_\alpha}$ are equivalent if $|\alpha|^{2\wedge}$ is non-zero on an open dense set. If $X = \mathbf{R}^n$ this holds in particular if α is of compact support or if $|\alpha|^2$ is of the form

$$|\alpha|^2(x) = k \exp\left\{-\sum_{r=1}^{n} \alpha_r x_r^2\right\}$$

where $\alpha_r > 0$ for $r = 1,\ldots, n$.

6. Some Properties of Covariant Instruments

In this section we study further the instruments already constructed.

Let us first consider the instruments whose observables are projection-valued measures, classified in Theorem 3. For simplicity of presentation we consider only the spin zero case.

If $G = \mathbf{R}^3 \circledS SO(3)$, $H = SO(3)$ and $X = \mathbf{R}^3$ then the system of imprimitity U, P of G, X induced by the trivial one-dimensional representation of H is associated in [11] with position measurements on a particle of positive mass and spin zero. In this case the instruments of Theorem 3 can be written in the form

$$\mathscr{E}(f, A) = \int_X f(x)\,\varphi(x)\,\mathrm{tr}[AP(dx)]$$

where $\varphi(\pi s) = Us\varphi Us^*$ and $\varphi \in \mathscr{T}_s(\mathscr{H})^+$ satisfies $\mathrm{tr}[\varphi] = 1$ and $\varphi(Uh) = (Uh)\,\varphi$ for all h in H. This equation shows that for such a particle and instrument the outgoing state is determined totally by the position distribution of the ingoing state. In particular the mean momentum of the outgoing state is independent of the momentum distribution of the ingoing state. This is not the sort of result one might anticipate on physical grounds. In particular it does not fit the simple model where a cosmic ray passes through a sensitised plane surface, leaving a mark at some point of the surface, and emerging in a slightly perturbed state. However it might be suitable to describe the model where an electron strikes and is absorbed by a plate at some point, from which a single secondary electron is immediately re-emitted.

We now give a much more detailed analysis of the class of covariant instruments constructed in Section 5, continuing with the notation developed there. We have to investigate the momentum distribution of the outgoing state and its relation to the momentum distribution of the ingoing state. The momentum observable Q is taken to be the projection-valued measure on the dual group $\hat{X}$ of X corresponding to the unitary representation U of X. Following the theorem in [4] which allows us to form the composition of two instruments, we see that there is a unique observable R_α on phase space $X \times \hat{X}$ such that for all Borel sets $E \subseteq X$ and $F \subseteq \hat{X}$ and all states $A \in \mathscr{T}_s(\mathscr{H})$

$$\begin{aligned}\operatorname{tr}[R_\alpha(E \times F)\, A] &= \operatorname{tr}(\mathscr{E}_\alpha(\chi_E, A)\, Q(F)] \\ &= \int_E \operatorname{tr}[P(x^{-1}\alpha)\, AP(x^{-1}\alpha)\, Q(F)]\, dx.\end{aligned}$$

We now give a very much more explicit form for the observable R_α, restricting attention to the case of spin zero, that is to the case where ρ is the trivial one-dimensional representation of H, so $\mathscr{H} = \mathscr{L}^2(X)$. It will turn out that the description will involve making use of the theory of over-complete families of states developed by MacKenna and Klauder in [14, 15]. We first review some of the results in the form we shall need them.

If $\gamma : X \times \hat{X} \to \{z \in C : |z| = 1\}$ is the natural coupling then the equation

$$V(y) = \int_X \overline{\gamma(x, y)}\, P(dx)$$

defines a strongly continuous unitary representation of $\hat{X}$ and the imprimitivity relations between U and P may be written in the form

$$(Ux)\,(Vy) = \overline{\gamma(xy)}\,(Vy)\,(Ux)$$

for all x in X and y in $\hat{X}$ (see [15]). Explicitly if $f \in \mathscr{L}^2(X)$ then

$$\{(Ux)f\}\,(z) = f(z + x)$$

and

$$\{(Vy)f\}\,(z) = \overline{\gamma(z, y)} f(z)$$

for almost all z in X. We suppose that the Haar measures of X and $\hat{X}$ are normalized so that the Fourier transform $f \to \hat{f}$ is an isometry of $\mathscr{L}^2(X)$ onto $\mathscr{L}^2(\hat{X})$.

Now recall that the function α defined in the last section lies in $\mathscr{L}^2(X)$ and define $\beta \to \beta' : \mathscr{L}^2(X) \to B(X \times \hat{X})$ by

$$\begin{aligned}\beta'(x, y) &= \langle \beta, Vy^* Ux^* \alpha \rangle \\ &= \int_X \overline{\gamma(t, y)}\, \overline{\alpha(t - x)}\, \beta(t)\, dt \\ &= \{P(x^{-1}\bar{\alpha})\, \beta\}^\wedge (y).\end{aligned}$$

It is clear that for all β in $\mathscr{L}^2(X)$, β' is a continuous bounded function with $|\beta'(x, y)| \leqslant \|\beta\|$, and it was shown in [14] that β' is in $\mathscr{L}^2(X \times \hat{X})$, the map $\beta \to \beta'$ being an isometric embedding of $\mathscr{H} = \mathscr{L}^2(X)$ into $\mathscr{L}^2(X \times \hat{X})$. Similarly we define a positive linear map $A \to A''$ of $\mathscr{T}_s(\mathscr{H})$ into $B(X \times \hat{X})$ by

$$A''(x, y) = \langle AVyUx^*\alpha, VyUx^*\alpha \rangle$$

and see that A'' is a continuous bounded function with $|A''(x, y)| \leqslant \|A\|$. If $A = \beta \otimes \bar{\beta}$ then $A''(x, y) = |\beta'(x, y)|^2$ and

$$\int_{X \times \hat{X}} A''(x, y)\, dx\, dy = \|\beta'\|^2 = \|\beta\|^2 = \mathrm{tr}[A].$$

By linearity this formula holds for all A in $\mathscr{T}_s(\mathscr{H})$ and A'' always lies in $\mathscr{L}^1(X \times \hat{X})$.

If $A = \beta \otimes \bar{\beta}$ and $E \subseteq X$ and $F \subseteq \hat{X}$ then

$$\begin{aligned}\int_{E \times F} A''(x, y)\, dx\, dy &= \int_{E \times F} |\langle \beta, Vy^*Ux^*\alpha \rangle|^2\, dy\, dx \\ &= \int_{x \in E} \left\{ \int_{y \in F} |\{P(x^{-1}\bar{\alpha})\, \beta\}^\wedge (y)|^2\, dy \right\} dx \\ &= \int_{x \in E} \langle Q(F)\{P(x^{-1}\bar{\alpha})\, \beta\}, \{P(x^{-1}\bar{\alpha})\, \beta\} \rangle\, dx \\ &= \int_E \mathrm{tr}[Q(F)\, P(x^{-1}\bar{\alpha})\, (\beta \otimes \bar{\beta})\, P(x^{-1}\alpha)]\, dx \\ &= \mathrm{tr}[R_\alpha(E \times F)\, A].\end{aligned}$$

We have only shown this for $A = \beta \otimes \bar{\beta}$ but by linearity it now holds for all A in $\mathscr{T}_s(\mathscr{H})$. By taking monotone limits, it follows that for all $E \subseteq X \times \hat{X}$ and all A in $\mathscr{T}_s(\mathscr{H})$

$$\mathrm{tr}[R_\alpha(E)\, A] = \int_E A''(x, y)\, dx\, dy.$$

This important equation may be presented in a slightly different form. Corresponding to the map $\beta \to \beta'$ of $\mathscr{H}$ into $\mathscr{H}'$ there is a positive isometric embedding $A \to A'$ of $\mathscr{T}_s(\mathscr{H})$ into $\mathscr{T}_s(\mathscr{H}')$ such that $(\beta \otimes \bar{\beta})' = \beta' \otimes \bar{\beta}'$ for all β in $\mathscr{H}$. There is also a projection-valued measure R' on $X \times \hat{X}$ to $\mathscr{T}_s(\mathscr{H}')$ given by

$$\langle R'(E) f, f \rangle = \int_{X \times \hat{X}} \chi_E \, |f(x, y)|^2 \, dx \, dy$$

for all Borel sets $E \subseteq X \times \hat{X}$ and all f in $\mathscr{L}^2(X \times \hat{X})$. We may now rewrite the above formula as

$$\mathrm{tr}[R_\alpha(E) A] = \mathrm{tr}[R'(E) A']$$

for all A in $\mathscr{T}_s(\mathscr{H})$ and all $E \subseteq X \times \hat{X}$. Thus we have obtained a particular case of a theorem of Neumark (see [16]) which asserts that any operator-valued probability measure on a Hilbert space can be regarded as the restriction of a projection-valued measure defined on a larger Hilbert space.

We are now in a position to investigate the properties of the instruments $\mathscr{E}_\alpha$ introduced in the last section and to see how much better behaved they are than the instruments whose observables are projection-valued measures. We have to determine the conditional probability distribution of momentum of any state A after it has passed through the instrument $\mathscr{E}_\alpha$. By virtue of the simple equality

$$\begin{aligned} \beta'(x, y) &= \int_X \overline{\gamma(t, y)} \, \overline{\alpha(t - x)} \, \beta(t) \, dt \\ &= \overline{\gamma(x, y)} \int_{\hat{X}} \gamma(x, s) \, \overline{\hat{\alpha}(s - y)} \, \hat{\beta}(s) \, ds \end{aligned}$$

we see that for all $A = \beta \otimes \bar{\beta}$ and all $F \subseteq \hat{X}$

$$\begin{aligned} \mathrm{tr}[\mathscr{E}_\alpha(X, A) Q(F)] &= \mathrm{tr}[R_\alpha(X \times F) A] \\ &= \int_{X \times F} |\beta'(x, y)|^2 \, dx \, dy \\ &= \int_{y \in F} \int_{x \in X} \left\{ \left| \int_{s \in \hat{X}} \gamma(x, s) \, \overline{\hat{\alpha}(s - y)} \, \hat{\beta}(s) \, ds \right|^2 \right\} dx \, dy \\ &= \int_{y \in F} \int_{s \in \hat{X}} | \overline{\hat{\alpha}(s - y)} \, \hat{\beta}(s)|^2 \, ds \, dy \\ &= \int_{\hat{X} \times \hat{X}} \chi_F(y) \, |\hat{\alpha}(s - y)|^2 \, |\hat{\beta}(s)|^2 \, dy \, ds \\ &= \int_{\hat{X}} (\chi_F \circ |\hat{\alpha}|^2)(s) \, |\hat{\beta}(s)|^2 \, ds \\ &= \mathrm{tr}[Q(\chi_F \circ |\hat{\alpha}|^2) \, \beta \otimes \bar{\beta}]. \end{aligned}$$

By the usual limiting arguments we now obtain

THEOREM 5. *If $\mathscr{E}_\alpha$ is the instrument constructed in Theorem 4 and $A \in \mathscr{T}_s(\mathscr{H})$ then the momentum distribution of the modified state $\mathscr{E}_\alpha(X, A)$ is given by the equation*

$$\mathrm{tr}[\mathscr{E}_\alpha(X, A)\, Q(f)] = \mathrm{tr}[Q(f \circ |\hat{\alpha}|^2)\, A]$$

for arbitrary f in $K(\hat{X})$. That is the momentum distribution of the outgoing state is the same as that of the ingoing state except for a perturbation due to the instrument $\mathscr{E}_\alpha$ and measured by the function $|\hat{\alpha}|^2$ on $\hat{X}$.

We have also shown that the observables R_α on phase space may be described as joint observables of position and momentum since the marginal observables with respect to the X-variable are modified position observables while the marginal observables with respect to the $\hat{X}$-variable are modified momentum observables.

Now let us specialize to the case where $X = \mathbf{R}^n$ and let us suppose that α satisfies for all $i, j = 1,\ldots, n$

$$\int_{\mathbf{R}^n} x_i \,|\alpha(x)|^2\, dx = 0; \qquad \int_{\mathbf{R}^n} y_j \,|\hat{\alpha}(y)|^2\, dy = 0;$$

$$\int_{\mathbf{R}^n} x_i^2 \,|\alpha(x)|^2\, dx = \lambda_i < \infty; \qquad \int_{\mathbf{R}^n} y_j^2 \,|\hat{\alpha}(y)|^2\, dy = \mu_j < \infty.$$

Then it is known (see [15]) that under suitable conditions on α the map $A \to A''$ is one-one so that all information about any A in $\mathscr{T}_s(\mathscr{H})$ can be derived by studying the "classical" probability distribution A'' in phase space. However as the inverse map is not continuous in any very natural sense the information may not be easy to obtain. If P_i and Q_j denote the usual position and momentum operators for the i-th position coordinate and j-th momentum coordinate respectively then the formulae we have already obtained for the marginal distributions tell us that for any vector state β for which the following expressions make sense

$$\langle \beta, \beta \rangle = \int_{\mathbf{R}^{2n}} |\beta'(x, y)|^2\, dx\, dy;$$

$$\langle P_i\beta, \beta \rangle = \int_{\mathbf{R}^{2n}} x_i\, |\beta'(x, y)|^2\, dx\, dy;$$

$$\langle Q_j\beta, \beta \rangle = \int_{\mathbf{R}^{2n}} y_j\, |\beta'(x, y)|^2\, dx\, dy;$$

$$\langle P_i^2\beta, \beta \rangle = \int_{\mathbf{R}^{2n}} x_i^2\, |\beta'(x, y)|^2\, dx\, dy - \lambda_i;$$

$$\langle Q_j^2\beta, \beta \rangle = \int_{\mathbf{R}^{2n}} y_j^2\, |\beta'(x, y)|^2\, dx\, dy - \mu_j\,.$$

That is the "classical" calculations made with respect to the probability distributions on phase space give the correct values for the mean position and momentum of any state and increase the variances of those quantities by constant amounts depending only on the parameter α of the instrument.

We come now to a general discussion of the problem of making simultaneous measurements of position and momentum in quantum mechanics. The rigorous form of Heisenberg's uncertainty principle is the mathematical equation

$$\{\langle P_i^2\beta, \beta\rangle - \langle P_i\beta, \beta\rangle^2\}\{\langle Q_i^2\beta, \beta\rangle - \langle Q_i\beta, \beta\rangle^2\} \geqslant \tfrac{1}{4}\hbar^2$$

which can be interpreted as meaning that in any experiment which measures both the position and the momentum of a state the product of the variances exceeds $\frac{1}{4}\hbar^2$. This is of course not in conflict with the results obtained from the observables R_α since all the variances calculated from them are slightly larger than their "true" values.

In one sense our observable does not give us a simultaneous measurement of position and momentum since it was constructed by first passing the state through an instrument with parameter α measuring position and then measuring the conditional momentum. However by virtue of the equations

$$\begin{aligned} |\beta'(x, y)| &= \left| \int_X \overline{\gamma(t, y)}\, \overline{\alpha(t - x)}\, \beta(t)\, dt \right| \\ &= \left| \int_{\hat{X}} \gamma(x, s)\, \hat{\alpha}(s - y)\, \hat{\beta}(s)\, ds \right| \end{aligned}$$

we see that exactly the same observable is obtained on phase space by first passing the state through an instrument with parameter $\hat{\alpha}$ measuring momentum and then measuring the conditional position (even though an experimental method of doing this is difficult to imagine). Therefore there is no way of telling from the observable R_α itself which of the measurements was done first and there is no reason why we should not regard R_α as an observable permitting simultaneous measurement of position and momentum, the parameter α representing an unavoidable perturbation the observable effects on the state in the process of measuring it.

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